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# DETERMINATION OF TURBULENT VELOCITY FIELD IN A RECTILINEAR DUCT WITH NON-CIRCULAR CROSS-SECTION

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**Abstract**—The equation of motion of a liquid flowing through a non-circular duct is solved by means of integral transformation. As a result of application of some integral transformation the problem of the turbulent flow has been brought to the solution of equation of the type describing the laminar flow. The solution of this equation is known and frequently cited in the literature of the subject. On the basis of these results the analytical expression describing the relation between the ratio $\nu_T/\nu_0$ and dimensionless velocity $u/u^*$, of rather general nature can be constructed. To illustrate the method some numerical example is included, too. The agreement between the computed velocity field and the experimental data are found to be quite satisfactory.

## NOMENCLATURE

$u$, longitudinal component of the velocity;
$u^*$, friction velocity $(\tau_w/\rho)^{\frac{1}{2}}$;
$u/u^*$, dimensionless velocity of the liquid;
$u_b$, bulk velocity;
$\nu_0$, molecular kinematic viscosity;
$\nu_T$, turbulent kinematic viscosity;
$\rho$, density of the liquid;
$\partial p/\partial x$, pressure gradient in the duct in the direction of the flow;
$f$, friction factor;
$\tau_w$, shear stress;
$r$, distance from the centre line of the duct;
$r_a$, outer radius of the duct;
$Y = r_a[1 - (r/r_a)]$, wall distance;
$Y^* = Y(u^*/\nu_0)$, dimensionless wall distance parameter;
$d$, hydraulic diameter.

## 1. INTRODUCTION

THE LITERATURE on turbulent flow has increased considerably in the course of years. Most of the existing analyses for turbulent flow are adequate only for a rectilinear duct of circular cross-section. Recently some authors have endeavoured to determine the turbulent velocity field in non-circular cross-section duct on the basis of experimental and analytical results for ducts of circular cross-section. Among authors dealing with this problem we can mention Deissler and Taylor [1], Buleev [2], Slykov and Carevski-Djakin [3].

The method applied by these authors is based on the assumption, that a regular contour may be transformed into circle in the new orthogonal coordinate system by conformal mapping. Then utilizing the well-known relations for round ducts the turbulent velocity fields can be determined. After the inverse transformation one obtains the velocity field for the given cross-section. The problem solved in this manner involves an enormous amount of computational work. The course of the computations presented here follows the hypothetical assumption, that the relation between the dimensionless distance from the wall $Y^*$ and the dimensionless velocity $u/u^*$ is of general nature and can be applied to the cross-section remarkably deviating in the shape from a circle.

In the present paper, the above assumption being taken valid, the dimensionless ratio $\nu_T/\nu_0$ was related directly to the dimensionless velocity $u/u^*$, thus eliminating the dimensionless distance $Y^*$ (as it is known the ratio $\nu_T/\nu_0$ and $u/u^*$ are functions of the parameter $Y^*$).

The ratio $\nu_T/\nu_0$ can be expanded in a power series of $u/u^*$ making use of the results and an analytical expression due to Reichardt.

Since the ratio $\nu_T/\nu_0$ depends in this case on the dimensionless velocity $u/u^*$ to be found, the differential equation of the motion becomes nonlinear.

This apparent difficulty of nonlinearity can be easily avoided by introducing a new function determined by means of some integral transformation. After introducing the new function the equation describing a turbulent flow becomes linear. The form of this equation of motion is identical with that for the laminar flow.

The function describing a turbulent velocity field can be easily obtained by algebraic manipulations. To illustrate the method a numerical example is also included.

## 2. FLUID FLOW ANALYSIS

The flow field for fully established turbulent flow is described by the following differential equation:

$$\operatorname{div}\left[\left(1+\frac{\nu_T}{\nu_0}\right)\operatorname{grad}\left(\frac{u}{u^*}\right)\right]=\frac{1}{\nu_0\rho u^*}\cdot\frac{\partial p}{\partial z}. \quad (1)$$

The symbols appearing in this and the following equations are contained in the Nomenclature. The liquid properties related to the present investigation will be assumed as constants. Because of its relatively simple mathematical form Reichardt's universal velocity distribution is considered. It can be written as:

$$\frac{u}{u^*}=2{\cdot}5\ln(1+0{\cdot}4\,Y^*)$$
$$+\,7{\cdot}8\left(1-\mathrm{e}^{-Y^*/11}-\frac{Y^*}{11}\,.\,\mathrm{e}^{-0{\cdot}33\,Y^*}\right);$$
$$Y^*\geqslant 0. \quad (2)$$

The relation determined by equation (2) as well as $\nu_T/\nu_0(Y^*)$ is presented in Fig. 1, taken from [4]. The ratio $\nu_T/\nu_0$ can be related directly to the dimensionless velocity $u/u^*$, thus eliminating the dimensionless distance parameter $Y^*$.

This relation is shown on the Fig. 2.

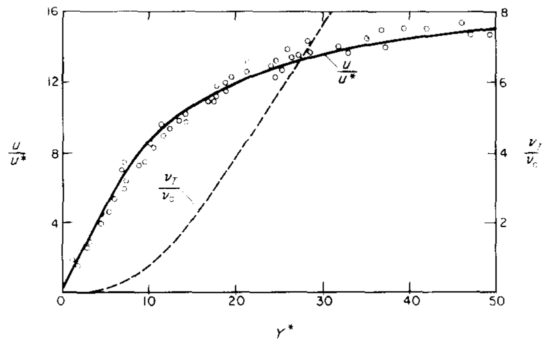


FIG. 1. Ratio of eddy viscosity to molecular viscosity $\nu_T/\nu_0$ and dimensionless velocity $u/u^*$ as function of the dimensionless distance from the wall, $Y^*$.

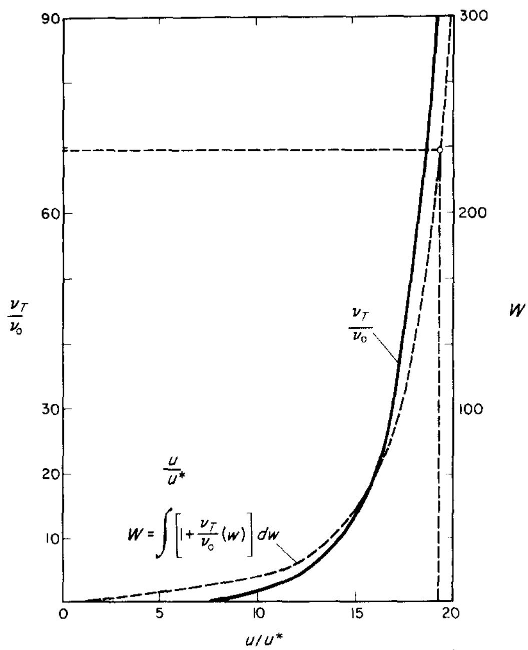


FIG. 2. Ratio of eddy viscosity to molecular viscosity $\nu_T/\nu_0$ and integral $W$ as function of the dimensionless velocity $u/u^*$.

Since the relation $v_T/v_0$ depends in this case on the dimensionless velocity $u/u^*$ to be found, the differential equation (1) of motion becomes nonlinear. This apparent difficulty of the nonlinearity can be avoided by introducing a new function determined by means of the integral transformation.

After introducing the integral transformation

$$W = \int_0^{u/u^*} \left[1 + \frac{v_T}{v_0}(w)\right] \mathrm{d}w \tag{3}$$

the equation (1) may be rewritten in the form:

$$\mathrm{div}\,(\mathrm{grad}\,W) = \frac{1}{v_0 \rho u^*} \cdot \frac{\partial p}{\partial z}. \tag{4}$$

The function $W$ could be evaluated by numerical or graphical integration according to the equation (3). The function $W(u/u^*)$ is shown on the Fig. 2. The equation (4) has a form identical with the equation for laminar flow.

The solution of this equation is known and frequently cited in the literature of the subject. To demonstrate the method a numerical example is presented.

### 3. NUMERICAL EXAMPLE

As an example we will determine the velocity field for a duct the cross-section of which has the shape of a narrow isosceles triangle with an apex angle of $\varphi = 11{\cdot}5$ degrees, corresponding to a side ratio 5:1. This case has been investigated experimentally in (3).

Introducing the dimensionless parameter

$$\frac{1}{v_0 \rho u^*} \cdot \frac{\partial p}{\partial z} = \alpha \quad \text{and} \quad \frac{W}{\alpha} = \overline{W}$$

the equation (4) then can be rewritten as:

$$\mathrm{div}\,(\mathrm{grad}\,\overline{W}) = -1. \tag{5}$$

The boundary condition for the $\overline{W}$ are:

$$\overline{W} = 0 \text{ on the contour.}$$

Approximate solution of equation (5) for triangle determined by straight lines: $y = \pm kx$, $x = h$ is:*

$$\overline{W} = \frac{(k^2x^2 - y^2)}{2(1 - k^2)} \cdot \left[1 - \left(\frac{x}{h}\right)^{v_1}\right] \tag{6}$$

where

$$v_1 = \frac{-4 + \sqrt{[6 + (10/k^2)]}}{2}.$$

The value taken for $\mathrm{tg}\,\varphi/2 = k$ in this case is:

$$\left.\begin{aligned} \mathrm{tg}\frac{\varphi}{2} &= k = 0{\cdot}1007 \\ k^2 &= 0{\cdot}01014. \end{aligned}\right\} \tag{7}$$

Introducing the value from (7) equation (6) becomes:

$$\overline{W} = 0{\cdot}506\,(0{\cdot}01014\,x^2 - y^2) \times \left[1 - \left(\frac{x}{h}\right)^{13{\cdot}8}\right]. \tag{8}$$

To compare the experimental velocity profile with the theoretical expression, equation (8) was used to calculate, for the velocity profiles measured on the straight lines $y = 0$, the ratio $W/W_{\max}$ as a function of $x/h$.

The equation then obtained can be written in the form:

$$\frac{W}{W_{\max}} = \frac{\overline{W}}{\overline{W}_{\max}} = 1{\cdot}545\left(\frac{x}{h}\right)^2 \left[1 - \left(\frac{x}{h}\right)^{13{\cdot}8}\right] \tag{9}$$

thereby

$$\overline{W}_{\max} = 0{\cdot}00333\,h^2$$

and

$$\left(\frac{x}{h}\right)_{W_{\max}} = 0{\cdot}86.$$

According to the definition the value of the dimensionless parameter $\alpha$ is given by:

$$\alpha = \frac{1}{v_0 \rho u^*} \cdot \frac{\partial p}{\partial z}. \tag{10}$$

* Equation (5) is solved by Kantorovich method in [6].

It is easy to derive that:

$$\alpha = F(Re, f, d). \tag{11}$$

According to a force balance over the flow section the pressure gradient is related to the friction factor by:

$$\frac{\partial p}{\partial z} = f \cdot \rho \cdot \frac{u_b^2}{2} \cdot \frac{4}{\alpha}. \tag{12}$$

The shear stress $\tau_w$ is related to the bulk velocity by the relation:

$$\tau_w = \tfrac{1}{2} \cdot f \cdot \rho \cdot u_b^2. \tag{13}$$

Besides, the friction velocity $u^*$ is defined by:

$$u^* = \sqrt{\left(\frac{\tau_w}{\rho}\right)} = u_b \cdot \sqrt{\left(\frac{f}{2}\right)}. \tag{14}$$

The equivalent hydraulic diameter $d$ associated with a triangle is given by the expression:

$$d = \frac{2h \cdot \sin \varphi}{(1 + \sin \varphi)} = \sqrt{(4h^2 \cdot 0{\cdot}008275)}. \tag{15}$$

The above equations can be substituted in equation (10). The equation then obtained can be written in the form:

$$\alpha \equiv \frac{1}{v_0 \sigma u^*} \cdot \frac{\partial p}{\partial z} = \frac{2 \cdot (\sqrt{2f}) \cdot Re}{d^2}. \tag{16}$$

From the measurement in the test channel it appeared that the ratio $f/f_B$ lies below unity:

$$f/f_B = 0{\cdot}85$$

where by $f_B$ is given with sufficient accuracy by the Blasius expression:

$$f_B = 0{\cdot}079\, Re^{-0{\cdot}25}. \tag{17}$$

In view of the above equations and with taking into account the relation (15) we have:

$$\alpha = \frac{69800}{h^2}. \tag{18}$$

The value taken for the $Re$ number in this case was: $Re = 10^4$.

The calculated value of $W_{max}$ is:

$$W_{max} = 0{\cdot}00333\, h^2 \alpha = 232. \tag{19}$$

*Table* 1.

| No. | $\frac{x}{h}$ | $\frac{W}{W_{max}}$ | $\left(\frac{u}{u_{max}}\right)_c$ | $\left(\frac{u}{u_{max}}\right)_m$ |
|---|---|---|---|---|
| 1 | 0·00 | 0·000 | 0·000 | 0·000 |
| 2 | 0·1 | 0·01545 | 0·184 | 0·19 |
| 3 | 0·2 | 0·0618 | 0·54 | 0·49 |
| 4 | 0·4 | 0·247 | 0·81 | 0·75 |
| 5 | 0·6 | 0·550 | 0·93 | 0·91 |
| 6 | 0·8 | 0·938 | 0·995 | 0·99 |
| 7 | 0·9 | 0·955 | 0·996 | 0·99 |
| 8 | 1·00 | 0·000 | 0·000 | 0·000 |

Table 1 gives the computation results of the velocity fields for various ratio $x/h$ and confronts the computation results with the experimental data. The confrontation is presented also in Fig. 3. The agreement between the computed velocity field and the experimental data are found to be quite satisfactory.

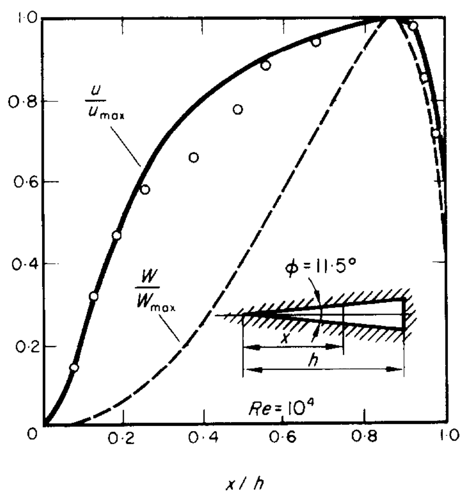


FIG. 3. Velocity distribution of turbulent flow $u/u_{max}$ in a triangle for various $x/h$ ratio for $Re = 10^4$.

## CONCLUSION

As a result of application of some integral transformation the problem of the turbulent flow has been reduced to the solution of equation of the type describing the laminar flow. The kind of approach utilized here essentially facilitates the consideration of turbulent flows and permits the use of well-known solutions, widely referred to in the literature.

In our calculations the experimental results of Reichardt and Nikuradse have been used. On this basis the analytical expression of rather general nature, describing the relation between the ratio $\nu_T/\nu_0$ and the dimensionless velocity $u/u^*$, has been constructed. This relation may be applied to the analysis of turbulent flow through ducts of complex geometry.

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## DÉTERMINATION D'UN CHAMP TURBULENT DE VITESSE DANS UNE CONDUITE RECTILIGNE À SECTION DROITE NON CIRCULAIRE


**Résumé**—On a effectué l'analyse de l'équation de mouvement pour le cas de l'écoulement d'un liquide dans un canal de section non-circulaire. Grace à l'utilisation d'une transformation integrale particulière on a ramené cette équation à la forme de l'équation différentielle, décrivant l'écoulement laminaire, dont la solution est connue. Les resultats obtenus permèttent espérer, que la dépendance de quantité $\nu_T/\nu_0$ en fonction de $u/u^*$ est de caractère general.

Pour illustrer la mèthode, on a calculé un exemple numérique. On a constaté une bonne concordance des calculs avec les données expérimentales.


## BESTIMMUNG DES TURBULENTEN GESCHWINDIGKEITSFELDES IN GERADLINIGEN KANÄLEN MIT NICHT-KREISFÖRMIGEM QUERSCHNITT


**Zusammenfassung**—Es wurde eine Analyse der Impulsgleichung für die turbulente Strömung, die in der geradlinigen Leitung von nicht kreisförmigem Querschnitt stattfindet, durchgeführt. Die erwähnte Gleichung wurde, durch Verwendung von spezieller Integraltransformation, auf die Differentialgleichung der laminaren Strömung zurückgeführt. Die Lösungen solcher Differentialgleichung sind sogar für komplizierte Geometrie des Querschnittes meistens bekannt.

Erhaltene Ergebnisse lassen vermuten, dass die funktionelle Abhängigkeit der Grösse $\nu_T/\nu_0$ von $u/u^*$ einen recht allgemeinen Charakter hat und ist von der Geometrie des Querschnittes unabhängig. Um den Rechnungsgang zu erläutern, wurde ein numerisches Beispiel durchgeführt. Die numerische Ergebnisse wurden mit denen eines Versuches verglichen.

Das analytische Ergebniss stimmt mit dem Versuchsergebniss weitgehend überein.


## ОПРЕДЕЛЕНИЕ ПРОФИЛЯ СКОРОСТИ ПРИ ТУРБУЛЕНТНОМ ТЕЧЕНИИ В КАНАЛАХ НЕКРУГЛОГО СЕЧЕНИЯ


**Аннотация**—Проведен анализ уравнения движения для турбулентного течения жидкости в канале некруглого сечения. Данное уравнение с помощью специального интегрального преобразования сводится к дифференциальному уравнению, описывающему ламинарное течение. Решение для такого рода уравнения известно в литературе. Полученные результаты позволяют предположить, что функциональная зависимость между величинами $\nu_T/\nu_0$ и $u/u^*$ имеют общий вид.

Для иллюстрации приводится численный пример. Обнаружено хорогее совпадение результатов расчета с экспериментальными данными.